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Optimal Sunny Selections for Metric Projections onto Unit Balls

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Optimal sunny selections of metric projections onto balls are determined for the normed spaces $C_p(Q)$ $(1 \leqslant p \leqslant \infty)$ and $L^1(\Omega, \mu)$, and their optimal Lipschitz constants are computed. Moreover, the uniqueness of the optimal sunny selection is proved for the Banach space $C(Q)$. © 1995 Academic Press, Inc.

1. INTRODUCTION

Let X be a real normed vector space of dimension greater than 1, and let C be a nonempty closed convex subset of X. Denote by $\mathscr{P}: X \to 2^C$ the *metric projection* onto C,

$$\mathscr{P}(x) = \{z \in C : \|x - z\| = \inf_{y \in C} \|x - y\|\}. \tag{1.1}$$

In general, it is possible that $\mathscr{P}$ is a multivalued mapping which is defined on a proper subset of X. Define the *optimal Lipschitz constant* of $\mathscr{P}$ by

$$K_{\mathscr{P}}(X) = \inf K_P(X),$$

where the infimum is taken over all selections P of $\mathscr{P}$ and $K_P(X)$ is the *best Lipschitz constant* of P defined by

$$K_P(X) = \sup \left\{ \frac{\|Px - Py\|}{\|x - y\|} : x \neq y \right\}.$$

Further, a metric selection T of $\mathscr{P}$ is said to be *optimal* if $K_T(X) = K_{\mathscr{P}}(X)$.

If C is equal to the unit ball

$$B = \{x \in X : \|x\| \leqslant 1\},$$

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then the *radial projection*

$$Rx = \begin{cases} x/\|x\|, & \text{if } x \notin B, \\ x, & \text{if } x \in B, \end{cases} \tag{1.2}$$

is a selection of the metric projection $\mathscr{P}$ defined on X such that $1 \leqslant K_R(X) \leqslant 2$. It was proved by de Figueiredo and Karlovitz [8] and by Thele [18] that identities $K_R(X) = 1$ and $K_R(X) = 2$ hold if and only if the Birkhoff's orthogonality is symmetric (this is equivalent to X being an inner-product space, whenever the dimension of X is greater than 2), and iff X is not uniformly non-square, respectively. Moreover, several other properties and estimates of $K_R(X)$ were established in [3–6, 9, 10, 14, 15]. Note also that optimal selections have applications in investigating the minimal displacement problem, retraction problem onto spheres [11, 12], and Fan's approximation principle for nonexpansive mapping [7, 14]. For example, it has been proved in [14] that there exists an optimal selection T of the metric projection onto the unit ball B of the Banach space L^∞ with the Lipschitz constant equal to 1, which enabled us to extend Fan's L^∞-approximation principle [7] as follows: For every nonexpansive mapping $F\colon B \to L^\infty$, there exists $x \in B$ such that

$$\|Fx - x\| = \inf_{y \in B} \|Fx - y\|.$$

In particular, Thele's result implies that $K_R(C(Q)) = 2$, where $C(Q)$ is the Banach space of all continuous real valued functions on a compact Hausdorff space Q equipped with the uniform norm

$$\|x\| = \|x\|_\infty = \sup_{s \in Q} |x(s)|.$$

On the other hand, Goebel and Komorowski [12] observed that the mapping $T\colon C(Q) \to B_\infty$ defined by

$$(Tx)(s) = \max\{-1, \min\{1, x(s)\}\}; \qquad x \in C(S), \quad s \in Q, \tag{1.3}$$

is an optimal selection of the metric projection $\mathscr{P}$ onto the unit ball

$$B_\infty = \{x \in C(Q) : \|x\|_\infty \leqslant 1\},$$

which has the best Lipschitz constant $K_T(C(Q))$ equal to 1. This optimal selection was applied in [11, 12] to construct retractions of $C(Q)$ onto the unit sphere with better Lipschitz constants than the constants which could be obtained by using the radial selections. In view of inequality (2.6) with $p = 2$, the selection T of $\mathscr{P}$ is called the *orthogonal projection* (*selection*).

In Section 2, we prove that the orthogonal projection T is also an optimal selection of the metric projection $\mathscr{P}$: $C_p(Q) \to 2^{B_\infty}$ ($C = B_\infty$ in (1.1)) which has the best Lipschitz constant $K_T(C_p(Q))$ equal to 1, whenever $1 \leqslant p < \infty$ and $C_p(Q)$ is the vector space $C(Q)$ with the L^p-norm

$$\|x\|_p = \left(\int_Q |x|^p \, d\mu \right)^{1/p}, \tag{1.4}$$

where μ denotes a positive Borel measure on Q. Moreover, we show that the optimal selection T of the metric projection $\mathscr{P}$: $C(Q) \to 2^{B_\infty}$ is unique in the class of all sunny selections P of $\mathscr{P}$.

In Section 3, we use orthogonal projections to determine the optimal selections and compute the optimal Lipschitz constants for the unit ball B_1 of the real Banach space $L^1(\Omega, \mu)$ of all μ-integrable functions (equivalence classes) on Ω, where (Ω, μ) is a positive measure space. In this case, by Thele's result we have again $K_R(L^1(\Omega, \mu)) = 2$. However, the optimal L^1-case is completely different from the optimal $C(Q)$-case. For example, we prove that $K_{\mathscr{P}}(L^1(\Omega, \mu)) < 2$ if and only if $L^1(\Omega, \mu)$ is a finite dimensional space.

2. Optimal Selections in $C_p(Q)$

Throughout this section, we assume that T is the orthogonal selection of the metric projection $\mathscr{P}$: $C(Q) \to B_\infty$. By (1.3) we have

$$Tx(s) = \begin{cases} sgn\, x(s), & \text{if } s \in M(x), \\ x(s), & \text{otherwise,} \end{cases} \tag{2.1}$$

where $sgn\, a = a/|a|$ if $a \neq 0$, $sgn\, 0 = 0$, and

$$M(x) = \{s \in Q : |x(s)| > 1\}. \tag{2.2}$$

Hence we get

$$Q \backslash M(x) = Z(x - Tx) := \{s \in Q : x(s) = Tx(s)\}. \tag{2.3}$$

Recall that a selection P of the metric projection $\mathscr{P}$: $C(Q) \to 2^{B_\infty}$ is said to be *sunny* [13] if

$$Px_\alpha = Px \tag{2.4}$$

for all $x \in C(Q)$ and $\alpha \geqslant 0$, where

$$x_\alpha = \alpha x + (1 - \alpha)\, Px. \tag{2.5}$$

THEOREM 2.1. *The orthogonal projection T is an optimal selection of the metric projection $\mathscr{P}$: $C_p(Q) \to 2^{B_\infty}$ for $1 \leqslant p \leqslant \infty$. Moreover, T is sunny and*

$$K_T(C_p(Q)) = K_{\mathscr{P}}(C_p(Q)) = 1.$$

Proof. The inequality

$$|a - sgn\, a| \leqslant |a - b|$$

holds for all real a and b such that $|a| \geqslant 1$ and $|b| \leqslant 1$. Hence one can insert $a = x(s)$ and $b = y(s)$, and use (2.1)–(2.3) to get

$$|x(s) - Tx(s)| \leqslant |x(s) - y(s)|$$

for all $s \in Q$, $x \in C(Q)$, and $y \in B_\infty$. This in conjunction with the monotonicity of the norm (1.4) yields

$$\|x - Tx\|_p \leqslant \|x - y\|_p \tag{2.6}$$

for all $y \in B_\infty$, i.e., T is a selection of the metric projection $\mathscr{P}$: $C_p(Q) \to 2^{B_\infty}$. Similarly, one can apply (2.1)–(2.3) together with the inequalities

$$|sgn\, a - sgn\, b| \leqslant |a - b|; \qquad |a|, |b| \geqslant 1,$$

and

$$|a - sgn\, b| \leqslant |a - b|; \qquad |a| \leqslant 1, |b| \geqslant 1,$$

to obtain

$$\|Tx - Ty\|_p \leqslant \|x - y\|_p$$

for all $x, y \in C(Q)$. Since $Tx = x$ on B_∞, it follows that T is optimal and $K_T(C_p(Q)) = 1$. Since T is identical with the single valued metric projection of the inner-product space $C_2(Q)$ onto the convex subset B_∞, it follows that T is sunny [13, 17]. This completes the proof. ∎

In the following, the symbol $\|\cdot\|$ denotes the uniform norm $\|\cdot\|_\infty$. Since Rx belongs to $\mathscr{P}(x)$, it follows from (1.2) that

$$\|x - Px\| = \|x - Rx\| = \|x\| - 1 \tag{2.7}$$

for all $x \in C(Q) \backslash B_\infty$ and $Px \in \mathscr{P}(x)$. Now, we can establish the main result of this section.

THEOREM 2.2. *A sunny optimal selection P of the metric projection $\mathscr{P}$: $C(Q) \to 2^{B_\infty}$ is unique, i.e., $P = T$.*

For the proof, note that the sunny optimal selection P satisfies (2.4) and the following characteristic inequalities:

$$\|x-Px\| \leqslant \|x-y\|, \qquad y \in B_{\mathscr{X}},$$

and

$$\|Px-Py\| \leqslant \|x-y\|; \qquad x, y \in C(Q). \tag{2.8}$$

Moreover, denote

$$E(x) = \{s \in Q : |x(s)| = \|x\|\}.$$

Since Q is compact, the set $E(x)$ is nonempty for every $x \in C(Q)$. Additionally, we have

$$Px(s) = sgn\ x(s), \tag{2.9}$$

whenever $s \in E(x)$ and $\|x\| > 1$. Indeed, by (2.7) and the fact that $|Px(s)| \leqslant 1$ we obtain

$$\|x\|-1 = \|x-Px\| \geqslant |x(s)-Px(s)| = |x(s)| - Px(s)\ sgn\ x(s).$$

Hence $Px(s)\ sgn\ x(s) \geqslant 1$, which gives (2.9). In the following three lemmas, it is assumed that P is a sunny optimal selection of $\mathscr{P}$: $C(Q) \to 2^{B_{\mathscr{X}}}$.

LEMMA 2.1. *If* $\|x\| > 1$ *then* $E(x) = E(x-Px)$.

Proof. If $s \in E(x)$ then by (2.7) we have

$$\|x\|-1 = \|x-Px\| \geqslant |x(s)-Px(s)| \geqslant \|x\|-1.$$

Hence we get $E(x) \subseteq E(x-Px)$. For an indirect proof of inclusion $E(x) \supseteq E(x-Px)$, we assume that $s \in E(x-Px) \backslash E(x)$ and $|x(s)| > 1$. Then one can use (2.7) and the fact that $|Px(s)| \leqslant 1$ to get

$$|x(s)| - Px(s)\ sgn\ x(s) = |x(s)-Px(s)| = \|x\|-1. \tag{2.10}$$

Next, we define $y \in C(Q)$ by

$$y(u) = \begin{cases} \dfrac{\|x\|+|x(s)|}{2}\ sgn\ x(u), & \text{if } |x(u)| \geqslant |x(s)|, \\[2ex] x(u) + \dfrac{\|x\|-|x(s)|}{2}\dfrac{x(u)}{|x(s)|}, & \text{otherwise.} \end{cases}$$

If $|x(u)| \geqslant |x(s)|$ then we have

$$|y(u)| = (\|x\| + |x(s)|)/2$$

and

$$|x(u) - y(u)| = ||x(u)| - (\|x\| + |x(s)|)/2| \leqslant (\|x\| - |x(s)|)/2.$$

Otherwise, we have

$$|y(u)| \leqslant |x(u)| + (\|x\| - |x(s)|)/2 \leqslant (\|x\| + |x(s)|)/2$$

and

$$|x(u) - y(u)| \leqslant (\|x\| - |x(s)|)/2,$$

where the last inequality can be replaced by the equality for $u = s$. Hence we obtain

$$\|y\| = |y(s)| = (\|x\| + |x(s)|)/2 > 1 \tag{2.11}$$

and

$$\|x - y\| = (\|x\| - |x(s)|)/2. \tag{2.12}$$

Therefore, by (2.9) we get

$$Py(s) = sgn\, y(s) = sgn\, x(s).$$

This together with (2.10) yields

$$\|Px - Py\| \geqslant |[Px(s) - Py(s)]\, sgn\, x(s)| = \|x\| - |x(s)|.$$

Since $s \notin E(x)$, it follows from (2.12) that

$$\|Px - Py\| > \|x - y\|,$$

which contradicts (2.8). Thus we have

$$|x(s)| = \|x\|, \tag{2.13}$$

whenever $x \in C(Q)$ is such that $s \in E(x - Px)$ and $|x(s)| > 1$. Finally, if $|x(s)| \leqslant 1$ and $s \in E(x - Px)$, then (2.7) gives

$$|x(s) - Px(s)| = \|x\| - 1 > 0.$$

Hence $|x_\alpha(s)| \to \infty$ as $\alpha \to \infty$. Choose $\alpha > 0$ so large that $|x_\alpha(s)| > 1$. Then (2.4) and (2.5) yield

$$|x_\alpha(s) - Px_\alpha(s)| = \alpha |x(s) - Px(s)| = \alpha \, \|x - Px\| = \|x_\alpha - Px_\alpha\|. \tag{2.14}$$

Thus $s \in E(x_\alpha - Px_\alpha)$, and we can apply (2.13) to get $|x_\alpha(s)| = \|x_\alpha\|$. Hence one can use (2.4) and (2.9) to derive

$$Px(s) = Px_\alpha(s) = sgn\, x_\alpha(s) = sgn[x_\alpha(s) - Px_\alpha(s)] = sgn[x(s) - Px(s)]$$

and

$$0 < |x(s) - Px(s)| = [x(s) - Px(s)]\, Px(s) = x(s)\, Px(s) - 1 \leqslant 0.$$

This contradiction completes the proof. ∎

LEMMA 2.2. *If* $\|x\| > 1$ *and* $\alpha \geqslant 0$, *then we have*

$$\|x_\alpha\| = \alpha \, \|x\| + 1 - \alpha.$$

Proof. Take an element $s \in E(x)$, and use (2.9) to get

$$\|x_\alpha\| \geqslant |x_\alpha(s)| = |\alpha x(s) + (1 - \alpha)\, sgn\, x(s)| = \alpha \, \|x\| + 1 - \alpha > 1.$$

Hence, as in (2.14), we conclude that $s \in E(x_\alpha - Px_\alpha)$. Thus Lemma 2.1 gives $\|x_\alpha\| = |x_\alpha(s)|$, which completes the proof. ∎

LEMMA 2.3. *We have* $sgn[Px(s)]\, sgn\, x(s) \geqslant 0$.

Proof. Without loss of generality, we assume that $\|x\| > 1$. If the desired inequality does not hold, then we have

$$sgn[Px(s)]\, sgn\, x(s) = -1 \tag{2.15}$$

and

$$-1 \leqslant -|Px(s)| = Px(s)\, sgn\, x(s) < 0. \tag{2.16}$$

By Lemma 2.2 and (2.5) it follows that

$$0 \leqslant \|x_\alpha\| + x_\alpha(s)\, sgn\, x(s) \to 1 - |Px(s)|,$$

as $\alpha \to 0$. Therefore, one can find a positive $\alpha < 1$ which is so small that

$$0 \leqslant (\|x_\alpha\| + x_\alpha(s)\, sgn\, x(s))/2 < 1$$

and

$$sgn\ x_\alpha(s) = sgn\ Px(s).$$

In particular, the last identity in conjunction with (2.15)–(2.16) yields

$$Px(s)\ sgn\ x(s) = -|Px(s)| < -|x_\alpha(s)| = x_\alpha(s)\ sgn\ x(s). \tag{2.17}$$

Next, define y in $C(Q)$ by

$$y(u) = \begin{cases} \dfrac{\|x_\alpha\|\ sgn\ x(s) + x_\alpha(s)}{2}, & \text{if } u \in A, \\ x_\alpha(u) + \dfrac{\|x_\alpha\|\ sgn\ x(s) - x_\alpha(s)}{2}, & \text{otherwise,} \end{cases}$$

where

$$A = \{u \in Q : x_\alpha(u)\ sgn\ x(s) \geqslant x_\alpha(s)\ sgn\ x(s)\}.$$

If $u \in A$ then we have

$$|y(u)| = (\|x_\alpha\| + x_\alpha(s)\ sgn\ x(s))/2$$

and

$$-\frac{\|x_\alpha\| - x_\alpha(s)\ sgn\ x(s)}{2} \leqslant x_\alpha(u)\ sgn\ x(s) - \frac{\|x_\alpha\| + x_\alpha(s)\ sgn\ x(s)}{2}$$
$$\leqslant \frac{\|x_\alpha\| - x_\alpha(s)\ sgn\ x(s)}{2}.$$

Otherwise, we get

$$-\frac{\|x_\alpha\| + x_\alpha(s)\ sgn\ x(s)}{2} \leqslant x_\alpha(u)\ sgn\ x(s) + \frac{\|x_\alpha\| - x_\alpha(s)\ sgn\ x(s)}{2}$$
$$\leqslant \frac{\|x_\alpha\| + x_\alpha(s)\ sgn\ x(s)}{2}$$

and

$$|x_\alpha(u) - y(u)| = (\|x_\alpha\| - x_\alpha(s)\ sgn\ x(s))/2.$$

By the first and third inequalities we obtain

$$\|y\| = (\|x_\alpha\| + x_\alpha(s)\ sgn\ x(s))/2 < 1.$$

Similarly, the second and fourth inequalities yield

$$\|x_\alpha - y\| = (\|x_\alpha\| - x_\alpha(s)\, sgn\, x(s))/2.$$

Hence it follows from the strict inequality (2.17) that

$$\begin{aligned} \|Py - Px_\alpha\| &\geqslant [y(s) - Px(s)]\, sgn\, x(s) \\ &= \frac{\|x_\alpha\| + x_\alpha(s)\, sgn\, x(s)}{2} - Px(s)\, sgn\, x(s) \\ &> \frac{\|x_\alpha\| - x_\alpha(s)\, sgn\, x(s)}{2} = \|y - x_\alpha\|, \end{aligned}$$

which contradicts (2.8). ∎

Proof of Theorem 2.2. In view of (2.1), we have to show that

$$Px(s) = sgn\, x(s), \qquad \text{if} \quad |x(s)| \geqslant 1,$$

and

$$Px(s) = x(s), \qquad \text{if} \quad |x(s)| < 1.$$

First, assume that

$$Px(s) \neq sgn\, x(s) \qquad \text{and} \qquad |x(s)| \geqslant 1.$$

Then by Lemma 2.3 we derive

$$0 \leqslant Px(s)\, sgn\, x(s) < 1 \qquad \text{and} \qquad |Px(s)| < 1.$$

Since we have

$$\begin{aligned} x_\alpha(s)\, sgn\, x(s) &= \alpha(x(s) - Px(s))\, sgn\, x(s) + Px(s)\, sgn\, x(s) \\ &= \alpha\, |x(s) - Px(s)| + Px(s)\, sgn\, x(s) \\ &> Px(s)\, sgn\, x(s) \geqslant 0, \end{aligned}$$

it follows that

$$sgn\, x(s) = sgn\, x_\alpha(s) \qquad \text{and} \qquad |Px(s)| < |x_\alpha(s)|, \tag{2.18}$$

whenever $\alpha > 0$. Moreover, by Lemma 2.2 and (2.5) we obtain $\|x_\alpha\| \to 1$, and $x_\alpha(s) \to Px(s)$, as $\alpha \to 0^+$. Hence there exists $\alpha > 0$ for which

$$(\|x_\alpha\| + x_\alpha(s)\, sgn\, x(s))/2 < 1. \tag{2.19}$$

Now define $y_\alpha \in C(Q)$ by

$$y_\alpha(u) = \begin{cases} \dfrac{\|x_\alpha\| + |x_\alpha(s)|}{2} \, sgn\, x_\alpha(u), & \text{if } |x_\alpha(u)| \geqslant |x_\alpha(s)|, \\ x_\alpha(u) + \dfrac{\|x_\alpha\| - |x_\alpha(s)|}{2} \dfrac{x_\alpha(u)}{|x_\alpha(s)|}, & \text{otherwise.} \end{cases}$$

Since y_α is defined exactly as the function y in the proof of Lemma 2.1, it follows from (2.11) and (2.12) that

$$\|y_\alpha\| = (\|x_\alpha\| + |x_\alpha(s)|)/2$$

and

$$\|x_\alpha - y_\alpha\| = (\|x_\alpha\| - |x_\alpha(s)|)/2.$$

This in conjunction with (2.18) and $\|y_\alpha\| < 1$ (see (2.19)) yields

$$\begin{aligned} \|Px_\alpha - Py_\alpha\| &\geqslant [y_\alpha(s) - Px_\alpha(s)] \, sgn\, x_\alpha(s) \\ &= \frac{\|x_\alpha\| + |x_\alpha(s)|}{2} - |Px(s)| \\ &> \frac{\|x_\alpha\| + |x_\alpha(s)|}{2} - |x_\alpha(s)| = \|x_\alpha - y_\alpha\|, \end{aligned}$$

which contradicts (2.8). Therefore, we have

$$Px(s) = sgn\, x(s), \tag{2.20}$$

whenever $|x(s)| \geqslant 1$. Finally, suppose that

$$Px(s) \neq x(s) \quad \text{and} \quad |x(s)| < 1.$$

Then we have

$$|x_\alpha(s)| > 1 \quad \text{and} \quad sgn\, x_\alpha(s) = sgn(x(s) - Px(s))$$

for sufficiently large $\alpha > 0$. Hence, by (2.4) and (2.20), we derive

$$|Px(s)| = |Px_\alpha(s)| = |sgn\, x_\alpha(s)| = 1.$$

Next, we apply Lemma 2.3 to get

$$\begin{aligned} 0 \leqslant sgn(x_\alpha(s))\, sgn(Px_\alpha(s)) &= sgn(x(s) - Px(s))\, sgn\, Px(s) \\ &= -sgn(Px(s))\, sgn\, Px(s) = -1, \end{aligned}$$

which leads to a contradiction and finishes the proof. ∎

3. Optimal Selections in $L^1(\Omega, \mu)$

First, we are going to construct the orthogonal selection onto the closed unit ball B_1 in the Banach space $L^1(\Omega, \mu)$ of all real valued μ-integrable functions (equivalence classes) defined on a positive measure space (Ω, μ) and equipped with the norm

$$\|x\| = \int_\Omega |x| \, d\mu.$$

For this purpose, we need the following elementary properties of the nondecreasing function

$$f(t) = \int_\Omega \min\{|x|, t\} \, d\mu, \qquad t \geqslant 0,$$

where $x \in L^1(\Omega, \mu)$.

Lemma 3.1. *The function f is a nondecreasing concave continuous function such that $f(0) = 0$ and $f(t) \to \|x\|$, as $t \to \infty$.*

Proof. If $|x(s)| \geqslant \lambda t_1 + (1-\lambda) t_2$ and $0 \leqslant \lambda \leqslant 1$, then we have

$$\begin{aligned} &\min\{|x(s)|, \lambda t_1 + (1-\lambda) t_2\} \\ &\quad = \lambda t_1 + (1-\lambda) t_2 \\ &\quad \geqslant \lambda \min\{|x(s)|, t_1\} + (1-\lambda) \min\{|x(s)|, t_2\}. \end{aligned}$$

Otherwise, we have

$$\begin{aligned} &\min\{|x(s)|, \lambda t_1 + (1-\lambda) t_2\} \\ &\quad = \lambda |x(s)| + (1-\lambda) |x(s)| \\ &\quad \geqslant \lambda \min\{|x(s)|, t_1\} + (1-\lambda) \min\{|x(s)|, t_2\}. \end{aligned}$$

By integrating these inequalities, we conclude that f is concave, and hence continuous on $(0, \infty)$. The functions

$$g_t(s) = \min\{|x(s)|, t\}, \qquad s \in \Omega,$$

belong to $L^1(\Omega, \mu)$ and $g_t(s) \downarrow 0$ pointwise, as $t \downarrow 0$. Hence the Monotone Convergence Theorem [1] implies that

$$f(t) = \int_\Omega g_t \, d\mu \to f(0) = 0, \qquad \text{as} \quad t \downarrow 0,$$

i.e., f is also continuous at $t=0$. Finally, to compute the limit of f at infinity, note that $f(t)=\|x\|$, whenever x is bounded almost everywhere on Ω and $t \geqslant |x|$ almost everywhere on Ω. Otherwise, it follows that

$$0 \leqslant |x(s)| - g_t(s) \downarrow 0 \qquad \text{almost everywhere, as} \quad t \uparrow \infty.$$

Hence one can apply the Monotone Convergence Theorem to get $f(t) \to \|x\|$ as $t \to \infty$, which completes the proof. ∎

By Lemma 3.1 the equation

$$\int_\Omega \min\{|x|, t\}\, d\mu = \|x\| - 1 \tag{3.1}$$

has the unique solution $t = t(x) > 0$ for each $x \in L^1(\Omega, \mu)$ with $\|x\| > 1$. Note that this equation can be rewritten in the following equivalent form

$$\int_{A_t(x)} |x - t\, sgn\, x|\, d\mu = 1, \tag{3.2}$$

where

$$A_t(x) = \{s \in \Omega : |x(s)| \geqslant t\}. \tag{3.3}$$

Now, let $t = t(x) > 0$ be the solution of equation (3.1), where $x \in L^1(\Omega, \mu)$ and $\|x\| > 1$. Then we define the mapping T by

$$Tx(s) = \begin{cases} x(s) - t\, sgn\, x(s), & \text{if} \quad s \in A_t(x), \\ 0, & \text{otherwise.} \end{cases} \tag{3.4}$$

Moreover, we put

$$Tx = x, \tag{3.5}$$

whenever $\|x\| \leqslant 1$.

By (3.2) and (3.4) it follows that $\|Tx\| = 1$, i.e., T is a projection onto the closed unit ball B_1. If $x \in L^1(\Omega, \mu) \cap L^2(\Omega, \mu)$ and $\|x\| > 1$, then (3.2)–(3.4) yield

$$\int_\Omega (x - Tx)(Tx - y)\, d\mu$$

$$= -\int_{\Omega \setminus A_t(x)} xy\, d\mu + \int_{A_t(x)} t\, sgn(x)(x - t\, sgn\, x - y)\, d\mu$$

$$= -\int_{\Omega \setminus A_t(x)} xy \, d\mu + t \int_{A_t(x)} |x - t \operatorname{sgn} x| \, d\mu$$

$$- t \int_{A_t(x)} y \operatorname{sgn} x \, d\mu$$

$$\geqslant t - t \left(\int_{\Omega \setminus A_t(x)} |y| \, d\mu + \int_{A_t(x)} |y| \, d\mu \right) = t(1 - \|y\|) \geqslant 0,$$

whenever $y \in B_1 \cap L^2(\Omega, \mu)$. By the well-known characterization of best approximations in an inner-product space by elements of convex sets, it follows that Tx is a best approximation to x by elements of the unit ball $B_1 \cap L^2(\Omega, \mu)$ in the inner-product space $L^1(\Omega, \mu) \cap L^2(\Omega, \mu)$ with L^2-norm. Therefore, the projection T: $L^1(\Omega, \mu) \to B_1$ is called the *orthogonal projection.* Clearly, its restriction

$$T\colon L^1(\Omega, \mu) \cap L^2(\Omega, \mu) \to B_1 \cap L^2(\Omega, \mu) \tag{3.6}$$

is sunny.

THEOREM 3.1. *The orthogonal projection T is a selection of the metric projection $\mathscr{P}$: $L^1(\Omega, \mu) \to 2^{B_1}$.*

Proof. By (3.2)–(3.4) we have

$$\|x - Tx\| = \int_{\Omega \setminus A_t(x)} |x| \, d\mu + \int_{A_t(x)} t \, d\mu$$

$$= \int_{\Omega} |x| \, d\mu - \int_{A_t(x)} |x - t \operatorname{sgn} x| \, d\mu$$

$$= \|x\| - 1 \leqslant \|x - y\|,$$

whenever $\|x\| > 1$ and $y \in B_1$. This completes the proof. ∎

An explicit formula for the orthogonal selection can be given in the special case of the Banach space l_n^1 $(n \geqslant 2)$ which consists of all real n-tuples $x = (x_1, ..., x_n)$ equipped with the norm

$$\|x\| = \sum_{k=1}^{n} |x_k|.$$

For a given $x \in l_n^1$ with $\|x\| > 1$, let $m(x) = (m_1, ..., m_n)$ be a rearrangement of

$$\Omega = \{1, ..., n\}$$

such that

$$|x_{m_1}| \geqslant |x_{m_2}| \geqslant \cdots \geqslant |x_{m_n}|. \tag{3.7}$$

Moreover, let $r = r(x)$ be the largest integer for which

$$r\,|x_{m_r}| \geqslant \sum_{i \in A} |x_i| - 1, \tag{3.8}$$

where

$$A = A(x) = \{m_1, ..., m_r\}. \tag{3.9}$$

Then by (3.7) we have

$$r\,|x_k| \geqslant \sum_{i \in A} |x_i| - 1, \qquad k \in A, \tag{3.10}$$

and

$$r\,|x_k| < \sum_{i \in A} |x_i| - 1, \qquad k \in \Omega \backslash A. \tag{3.11}$$

Indeed, if (3.11) is not satisfied, then we obtain

$$(r+1)\,|x_{m_{r+1}}| \geqslant \sum_{i \in A} |x_i| - 1 + |x_{m_{r+1}}|,$$

which contradicts the definition of r. In the following, we denote

$$Tx = (Tx_1, ..., Tx_n)$$

for $x \in l_n^1$.

COROLLARY 3.1. *The orthogonal selection T of the metric projection $\mathscr{P}\colon l_n^1 \to 2^{B_1}$ is given on $l_n^1 \backslash B_1$ by the formula*

$$Tx_k = \begin{cases} x_k - \dfrac{\sum_{i \in A} |x_i| - 1}{r}\, sgn\, x_k, & \text{if } k \in A, \\ 0, & \text{if } k \in \Omega \backslash A, \end{cases}$$

where $r = r(x)$ and $A = A(x)$ are defined by (3.7)–(3.9).

Proof. Let μ be the counting measure on $\Omega = \{1, 2, ..., n\}$, and let

$$t = \left(\sum_{i \in A} |x_i| - 1\right) \Big/ r.$$

Then t satisfies equation (3.2). Indeed, by (3.10) and (3.11), we have $t>0$ and

$$\sum_{k \in A} |x_k - tsgn\, x_k| = \sum_{k \in A} (|x_k| - t) = 1,$$

which completes the proof. ∎

As in the case of $C(Q)$ space, the orthogonal selection $T: l_n^1 \to B_1$ is optimal.

THEOREM 3.2. *The orthogonal projection T is an optimal selection of the metric projection $\mathscr{P}: l_n^1 \to 2^{B_1}$. Moreover, T is sunny and*

$$K_T(l_n^1) = K_{\mathscr{P}}(l_n^1) = \frac{2(n-1)}{n}.$$

For the proof we need the following two lemmas.

LEMMA 3.2. *If* $x = (1/(n-1), ..., 1/(n-1))$ *and* $y = (1/(n-1), ..., 1/(n-1), 0)$ *are elements of l_n^1, then we have*

$$\|Tx - Ty\| = \frac{2(n-1)}{n} \|x - y\|.$$

Proof. Since $\|y\| = 1$, we have $Ty = y$. Moreover, we have $r(x) = n$ and $A(x) = \Omega$ in (3.8) and (3.9). Hence, by Corollary 3.1, we get

$$Tx_k = \frac{1}{n}, \qquad k = 1, ..., n.$$

Therefore, we have

$$\|Tx - Ty\| = \frac{2}{n} \qquad \text{and} \qquad \|x - y\| = \frac{1}{n-1},$$

which completes the proof. ∎

LEMMA 3.3. *The inequality*

$$\|Tx - Ty\| \leqslant \frac{2(n-1)}{n} \|x - y\|$$

holds for all $x, y \in l_n^1$.

Proof. Let x and y ($\|x\| > 1$) be arbitrary elements in l_n^1. Without loss of generality, we assume that coordinates of x (and y) are arranged and their signs are changed in such (the same) way that

$$x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n \geqslant 0. \tag{3.12}$$

Note that

$$A = A(x) = \{1, ..., r\}$$

for some r ($1 \leqslant r \leqslant n$), and that

$$Tx_k \geqslant 0, \qquad k \in A, \tag{3.13}$$

which follows immediately from (3.10) and Corollary 3.1. Moreover, we have

$$\sum_{k=r+1}^{n} |x_k| \leqslant \frac{n-r}{n} d, \tag{3.14}$$

where $d = \|x\| - 1$ and the left hand side is equal to 0 for $r = n$. Indeed, by taking the sum of inequalities (3.11), we derive

$$r \sum_{k=r+1}^{n} |x_k| < (n-r)\left(\sum_{i=1}^{r} |x_i| - 1\right).$$

Hence we get

$$n \sum_{k=r+1}^{n} |x_k| < (n-r)\left(\sum_{i=1}^{n} |x_i| - 1\right),$$

which finishes the proof of (3.14). We denote by $\alpha = card\, B$ the number of elements of the set

$$B = \{k \in A : Tx_k \geqslant y_k\}.$$

Note that $\alpha \geqslant 1$, whenever $\|y\| \leqslant 1$. Indeed, if $B = \phi$ then, by (3.13) we get $y_k > Tx_k \geqslant 0$ ($k = 1, ..., r$) and $1 \geqslant \|y\| > \|Tx\| = 1$, a contradiction. Now, denote

$$t = \left(\sum_{i=1}^{r} x_i - 1\right) \Big/ r,$$

and suppose first that $\|y\| \leqslant 1$. Then apply Corollary 3.1 together with (3.5) and (3.12)–(3.14) to get

$$\begin{aligned}
\|Tx-Ty\| &= \|Tx-y\| \\
&= \sum_{k\in B}(x_k-t-y_k)+\sum_{k\in A\setminus B}(y_k-x_k+t)+\sum_{k=r+1}^{n}|y_k| \\
&= \left[\sum_{k\in B}(x_k-y_k)+\sum_{k\in A\setminus B}(y_k-x_k)+\sum_{k=r+1}^{n}(|y_k|-x_k)\right] \\
&\quad+\sum_{k=r+1}^{n}x_k-\frac{\alpha}{r}\left(d-\sum_{k=r+1}^{n}x_k\right)+\frac{r-\alpha}{r}\left(d-\sum_{k=r+1}^{n}x_k\right) \\
&\leqslant \|x-y\|+\left(1-\frac{2\alpha}{r}\right)d+\frac{2\alpha}{r}\sum_{k=r+1}^{n}x_k \\
&\leqslant \|x-y\|+\left(1-\frac{2\alpha}{r}\right)d+\frac{2\alpha}{r}\frac{n-r}{n}d \\
&= \|x-y\|+\left(1-\frac{2\alpha}{n}\right)(\|x\|-1).
\end{aligned}$$

Hence we derive

$$\|Tx-Ty\|\leqslant\|x-y\|,$$

whenever $n\leqslant 2\alpha$. Otherwise, we have

$$\begin{aligned}
\|Tx-Ty\| &\leqslant \|x-y\|+\left(1-\frac{2\alpha}{n}\right)\|x-y\| \\
&= \frac{2(n-\alpha)}{n}\|x-y\|\leqslant\frac{2(n-1)}{n}\|x-y\|,
\end{aligned}$$

which completes the proof when $\|y\|\leqslant 1$.

Thus it remains to consider the case when $\|y\|>1$. Without loss of generality, x and y can be interchanged. Therefore, in addition to (3.12), we assume that

$$\sum_{i\in A}x_i\geqslant\sum_{i\in A}|y_i|, \tag{3.15}$$

where $A=A_1\cup A_2$, $A_1=A(x)=\{1,...,r\}$, and the set $A_2=A(y)=\{m_1,...,m_p\}$ with $p=r(y)$ is defined by formulae (3.7) and (3.8), in which x is replaced by y. Denote

$$C=A_1\cap A_2, \qquad D=\{k\in C: Tx_k\geqslant Ty_k\},$$

$$\alpha=card\,D, \qquad d_1=\sum_{i\in A_1}x_i-1, \quad \text{and} \quad d_2=\sum_{i\in A_2}|y_i|-1.$$

Then we have $C=\{m_1, ..., m_q\}$, $0 \leqslant \alpha \leqslant q \leqslant \min\{p, r\}$, $A \backslash A_1 = A_2 \backslash C$, and $A \backslash A_2 = A_1 \backslash C$. Since inequalities (3.11) give

$$r x_k < d_1 \qquad \text{for} \quad k \in A_2 \backslash C,$$

we get

$$r \sum_{k \in A_2 \backslash C} x_k \leqslant (p-q)\, d_1 = (p-q) \left(\sum_{k \in A} x_k - 1 - \sum_{k \in A \backslash A_1} x_k \right).$$

Hence we have

$$n_o \sum_{k \in A_2 \backslash C} x_k \leqslant (n_o - r)\, c_1, \tag{3.16}$$

where $n_o = r + p - q$ and

$$c_1 = \sum_{k \in A} x_k - 1 = d_1 + \sum_{k \in A \backslash A_1} x_k. \tag{3.17}$$

Similarly, we use inequalities

$$p\, |y_k| < d_2 \qquad \text{for} \quad k \in A_1 \backslash C,$$

in order to get

$$n_o \sum_{k \in A_1 \backslash C} |y_k| \leqslant (n_o - p)\, c_2, \tag{3.18}$$

where

$$c_2 = \sum_{k \in A} |y_k| - 1 = d_2 + \sum_{k \in A \backslash A_2} |y_k|. \tag{3.19}$$

Now, by Corollary 3.1 we obtain

$$\begin{aligned} \|Tx - Ty\| = & \sum_{k \in A_1 \backslash C} \left(x_k - \frac{d_1}{r} \right) + \sum_{k \in C} \left| x_k - \frac{d_1}{r} - y_k + \frac{d_2}{p} \, sgn\, y_k \right| \\ & + \sum_{k \in A_2 \backslash C} \left(|y_k| - \frac{d_2}{p} \right) \end{aligned}$$

$$= \sum_{k \in A_1 \setminus C} (x_k - |y_k|) + \sum_{k \in A_1 \setminus C} |y_k| - (r-q)\frac{d_1}{r}$$
$$+ \sum_{k \in D} (x_k - y_k) - \alpha \frac{d_1}{r} + \sum_{k \in D} \frac{d_2}{p} sgn\, y_k$$
$$+ \sum_{k \in C \setminus D} (y_k - x_k) + \sum_{k \in C \setminus D} \left(\frac{d_1}{r} - \frac{d_2}{p} sgn\, y_k\right)$$
$$+ \sum_{k \in A_2 \setminus C} (|y_k| - x_k) + \sum_{k \in A_2 \setminus C} x_k - (p-q)\frac{d_2}{p}.$$

If $k \in C \setminus D$, then it follows from (3.10) that

$$0 \leqslant Tx_k < Ty_k = \left(|y_k| - \frac{d_2}{p}\right) sgn\, y_k \qquad \text{and} \qquad |y_k| - \frac{d_2}{p} \geqslant 0.$$

Hence we have

$$sgn\, y_k = 1 \qquad \text{for} \quad k \in C \setminus D.$$

This in conjunction with (3.16)–(3.19) yields

$$\|Tx - Ty\| \leqslant \sum_{k \in A} |x_k - y_k| + \sum_{k \in A_1 \setminus C} |y_k| - (r-q)\frac{d_1}{r} - \alpha\frac{d_1}{r} + \alpha\frac{d_2}{p}$$
$$+ (q-\alpha)\left(\frac{d_1}{r} - \frac{d_2}{p}\right) + \sum_{k \in A_2 \setminus C} x_k - (p-q)\frac{d_2}{p}$$
$$\leqslant \|x-y\| + \frac{d_1}{r}(2q - 2\alpha - r) + \sum_{k \in A_2 \setminus C} x_k$$
$$+ \frac{d_2}{p}(2\alpha - p) + \sum_{k \in A_1 \setminus C} |y_k|$$
$$= \|x-y\| + \frac{c_1}{r}(2q - 2\alpha - r) + \frac{2}{r}(r + \alpha - q) \sum_{k \in A_2 \setminus C} x_k$$
$$+ \frac{c_2}{p}(2\alpha - p) + \frac{2}{p}(p - \alpha) \sum_{k \in A_1 \setminus C} |y_k|$$
$$\leqslant \|x-y\| + \frac{c_1}{r}(2q - 2\alpha - r) + \frac{2c_1}{r}(r + \alpha - q)\frac{n_o - r}{n_o}$$
$$+ \frac{c_2}{p}(2\alpha - p) + \frac{2c_2}{p}(p - \alpha)\frac{n_o - p}{n_o}$$
$$= \|x-y\| + \frac{r + 2\alpha - p - q}{n_o}(c_2 - c_1).$$

To complete the proof, it remains to show that

$$\frac{r+2\alpha-p-q}{n_o}(c_2-c_1)\leqslant\frac{n-2}{n}\|x-y\|. \tag{3.20}$$

Note that $1\leqslant n_o=r+p-q\leqslant n$. Moreover, by (3.15) we have $c_1\geqslant c_2$. Hence the inequality is true when $r+2\alpha-p-q\geqslant 0$. Otherwise, by (3.17) and (3.19) we get

$$c_1-c_2=\sum_{i\in A}(x_i-|y_i|)\leqslant\|x-y\|. \tag{3.21}$$

Additionally, the inequality

$$p+q-r-2\alpha=2(p-\alpha)-n_o\leqslant n_o-2 \tag{3.22}$$

holds if and only if

$$p-\alpha+1\leqslant n_o.$$

The last inequality is obvious when $p<n_o$. Otherwise, we have $n_o=p\geqslant q=r$, and so

$$C=A_1=\{1,...,r\} \qquad \text{and} \qquad A_2=\{1,...,r,m_{r+1},...,m_p\}.$$

This in conjunction with (3.13) and Corollary 3.1 yields

$$\sum_{k=1}^{r}Tx_k=\|Tx\|=1=\sum_{k\in A_2}|Ty_k|\geqslant\sum_{k=1}^{r}|Ty_k|,$$

which is possible only when $Tx_k\geqslant Ty_k$ for some k with $1\leqslant k\leqslant r$. This means that $D\neq\phi$, i.e., $\alpha\geqslant 1$. Hence the inequality $p-\alpha+1\leqslant n_o$ is also true in the case when $p=n_o$, which completes the proof of (3.22). By (3.21) and (3.22), the proof of the first inequality in (3.20) is completed. ∎

Proof of Theorem 3.2. Let T be the orthogonal selection of the metric projection $\mathscr{P}\colon l_n^1\to 2^{B_1}$. Then the sunny property of T follows immediately from (3.6). Moreover, by Lemmas 3.2 and 3.3 we have

$$K_T(l_n^1)=\frac{2(n-1)}{n}.$$

Hence it remains to find an element $x\in l_n^1\backslash B_1$ such that, for every $z=Px\in\mathscr{P}(x)$, there exists y which satisfies $\|y\|=1$ and

$$\|z-y\|\geqslant\|Tx-y\|=\frac{2(n-1)}{n}\|x-y\|. \tag{3.23}$$

For this purpose, put

$$x=\left(\frac{1}{n-1}, \ldots, \frac{1}{n-1}\right) \in l_n^1 \quad \text{and} \quad y^i = x - \frac{1}{n-1} e_i,$$

where e_i is the unit vector in l_n^1 with its ith coordinate equal to 1. It is clear that $\|y^i\| = 1$. Moreover, by the proof of Lemma 3.2 we have $Tx_k = 1/n$ for $k = 1, \ldots, n$. Hence we easily compute that

$$\|Tx - y^i\| = \frac{2}{n} \quad \text{and} \quad \|x - y^i\| = \frac{1}{n-1}, \tag{3.24}$$

which proves the identity in (3.23) for $y = y^i$ $(i = 1, \ldots, n)$.

To construct the required y, note that the assumption $z \in \mathscr{P}(x)$ implies that $z_i \geqslant 0$, $\|z\| = 1$, and $z_j \geqslant 1/n$ for some $j \in \Omega = \{1, \ldots, n\}$. Moreover, we denote

$$A_3 = \Omega \backslash A_1 \backslash A_2 \backslash \{j\},$$

where

$$A_1 = \left\{ i \in \Omega \backslash \{j\} : z_i \leqslant \frac{1}{n} \right\} \quad \text{and} \quad A_2 = \left\{ i \in \Omega \backslash A_1 \backslash \{j\} : \frac{1}{n} < z_i \leqslant \frac{1}{n-1} \right\}.$$

If we put $c_i = card\, A_i$ $(i = 1, 2, 3)$, then we get

$$c_1 + c_2 + c_3 + 1 = n, \qquad z_j + \sum_{i \in A_3} z_i = 1 - \sum_{i \in A_1 \cup A_2} z_i,$$

and

$$\sum_{i \in A_1 \cup A_2} z_i + \frac{c_3}{n-1} + \frac{1}{n} \leqslant \sum_{i \in A_1 \cup A_2} z_i + \sum_{i \in A_3} z_i + z_j = 1.$$

Hence it follows that

$$\begin{aligned}
\|z - y^j\| &= z_j + \sum_{i \neq j} \left| z_i - \frac{1}{n-1} \right| \\
&= z_j + \sum_{i \in A_1} \left(\frac{1}{n-1} - z_i \right) + \sum_{i \in A_2} \left(\frac{1}{n-1} - z_i \right) + \sum_{i \in A_3} \left(z_i - \frac{1}{n-1} \right) \\
&= z_j - \sum_{i \in A_1 \cup A_2} z_i + \sum_{i \in A_3} z_i + \frac{c_1 + c_2 - c_3}{n-1}
\end{aligned}$$

$$= 1 - 2 \sum_{i \in A_1 \cup A_2} z_i + \frac{n-1-2c_3}{n-1}$$

$$= \frac{2}{n} + 2\left(1 - \sum_{i \in A_1 \cup A_2} z_i - \frac{c_3}{n-1} - \frac{1}{n}\right) \geqslant \frac{2}{n}$$

This together with (3.24) gives the inequality in (3.23) for $y = y^j$. Hence the proof is finished. ∎

Now, we show the optimality of the orthogonal selection T of the metric projection $\mathscr{P}$: $L^1(\Omega, \mu) \to 2^{B_1}$ in the case when the Banach space $L^1(\Omega, \mu)$ is infinite dimensional. Since $Tx \in \mathscr{P}(x)$, we have

$$\|x - Tx\| \leqslant \|x - y\|$$

for all $y \in B_1$. By the triangle inequality and the fact that $Ty = y$, it follows that

$$\|Tx - Ty\| \leqslant 2\ \|x - y\|, \tag{3.25}$$

whenever $x, y \in L^1(\Omega, \mu)$, $\|x\| > 1$, and $\|y\| \leqslant 1$.

THEOREM 3.3. *Let the Banach space $L^1(\Omega, \mu)$ be infinite dimensional. Then the orthogonal projection T is an optimal selection of the metric projection $\mathscr{P}$: $L^1(\Omega, \mu) \to 2^{B_1}$. Moreover, T is sunny and*

$$K_T(L^1(\Omega, \mu)) = K_{\mathscr{P}}(L^1(\Omega, \mu)) = 2.$$

For the proof, recall that $t = t(x) > 0$ denotes the unique solution of the equation

$$\int_\Omega \min\{|x|, t\}\ d\mu = \|x\| - 1, \tag{3.26}$$

whenever $\|x\| > 1$. Moreover, extend $t(x)$ to the unit ball by setting

$$t(x) = 0, \qquad x \in B_1. \tag{3.27}$$

Then the orthogonal selection can be written in the form

$$Tx = (x - t(x)\ sgn(x))\ \chi_{A(x)}, \qquad x \in L^1(\Omega, \mu),$$

where $\chi_{A(x)}$ denotes the characteristic function of the set

$$A(x) = \{s \in \Omega : |x(s)| \geqslant t(x)\}.$$

The function $x \to t(x)$ has the following nice properties.

LEMMA 3.4. *The function* $x \to t(x)$, $x \in L^1(\Omega, \mu)$, *is a convex continuous function which satisfies*

$$0 \leqslant t(|x|) = t(x) \leqslant t(y),$$

whenever $|x| \leqslant |y|$.

Proof. Note that the constant $t(x)$ is integrable on $A(x)$, and so $\mu(A(x)) < \infty$, whenever $\|x\| > 1$. Now, suppose that x, $y \geqslant 0$, $\|x\| > 1$, $0 < \lambda < 1$, and $x_\lambda = \lambda x + (1-\lambda)\, y \notin B_1$. Then we have

$$\begin{aligned}
&\int_\Omega \min\{x_\lambda, t(x_\lambda)\}\, d\mu \\
&\quad = \|x_\lambda\| - 1 \leqslant \lambda(\|x\| - 1) + (1-\lambda)(\|y\| - 1) \\
&\quad \leqslant \lambda \int_\Omega \min\{x, t(x)\}\, d\mu + (1-\lambda) \int_\Omega \min\{y, t(y)\}\, d\mu \\
&\quad \leqslant \int_\Omega \min\{x_\lambda, \lambda t(x) + (1-\lambda)\, t(y)\}\, d\mu.
\end{aligned}$$

Since the function

$$t \to \int_\Omega \min\{x_\lambda, t\}\, d\mu$$

is nondecreasing, it follows that

$$t(x_\lambda) \leqslant \lambda t(x) + (1-\lambda)\, t(y).$$

In view of (3.27), this inequality is also true when either $x_\lambda \in B_1$, or $x, y \in B_1$. Hence the function $x \to t(x)$, $x \geqslant 0$, is convex. Clearly, if $x \in B_1$ then $t(x) \leqslant t(y)$ for all y. Further, suppose that $0 \leqslant x \leqslant y$ and $\|x\| > 1$. If $t(x) > t(y)$ then one can use (3.2) together with $A(x) \subseteq A(y)$ to get

$$1 = \int_{A(x)} (x - t(x))\, d\mu < \int_{A(y)} (y - t(y))\, d\mu = 1.$$

Therefore, we have $t(x) \leqslant t(y)$, whenever $0 \leqslant x \leqslant y$. Since, by (3.26)–(3.27), we have $t(|x|) = t(x)$, it follows that

$$\begin{aligned}
t(\lambda x + (1-\lambda)\, y) &= t(|\lambda x + (1-\lambda)\, y|) \\
&\leqslant t(\lambda\, |x| + (1-\lambda)\, |y|) \leqslant \lambda t(x) + (1-\lambda)\, t(y)
\end{aligned}$$

for all $x, y \in L^1(\Omega, \mu)$ and $\lambda \in (0, 1)$. Thus the function $x \to t(x)$ is convex on $L^1(\Omega, \mu)$. Moreover, we have $t(B_1) = \{0\}$. Therefore, in view of Theorem 1.3 [2, p. 90], the function $x \to t(x)$ is continuous on $L^1(\Omega, \mu)$, which completes the proof of the lemma. ∎

Proof of Theorem 3.3. First we prove continuity of the orthogonal selection T. For this purpose, note that the formula (3.4) directly yields $T(|x|) = |Tx|$. Hence it is sufficient to prove continuity of T only for $x \geqslant 0$. For this purpose, suppose that $x \geqslant 0$ and $x_n \to x$ in $L^1(\Omega, \mu)$. In view of (3.25), we can assume that $\|x\| > 1$ and $\|x_n\| > 1$. Then by (3.4) we get

$$\begin{aligned} \|Tx - T(x_n)\| \leqslant & \int_{A(x) \cap A(x_n)} |x - t(x) - x_n + t(x_n)| \, d\mu \\ & + \int_{(\Omega \setminus A(x_n)) \cap A(x)} (x - t(x)) \, d\mu \\ & + \int_{(\Omega \setminus A(x)) \cap A(x_n)} (|x_n| - t(x_n)) \, d\mu. \end{aligned} \tag{3.28}$$

Next, take ε such that $0 < \varepsilon < t(x)$. Since $\|x_n\| \to \|x\|$ and $t(x_n) \to t(x)$, there exists an integer n_ε such that

$$\|x_n\| \leqslant \|x\| + \varepsilon \qquad \text{and} \qquad t(x) - \varepsilon \leqslant t(x_n) \leqslant t(x) + \varepsilon$$

for every $n \geqslant n_\varepsilon$. If $n \geqslant n_\varepsilon$ then we have

$$x(s) - t(x) \leqslant x(s) - t(x_n) + \varepsilon < x(s) - x_n(s) + \varepsilon \leqslant |x(s) - x_n(s)| + \varepsilon,$$

whenever $s \in (\Omega \setminus A(x_n)) \cap A(x)$, and

$$|x_n(s)| - t(x_n) \leqslant |x_n(s)| - t(x) + \varepsilon < |x_n(s)| - x(s) + \varepsilon \leqslant |x(s) - x_n(s)| + \varepsilon$$

for all $s \in (\Omega \setminus A(x)) \cap A(x_n)$. Additionally, we have

$$\begin{aligned} (t(x) - \varepsilon) \, \mu(A(x_n)) & \leqslant \int_{A(x_n)} t(x_n) \, d\mu \\ & \leqslant \int_{A(x_n)} |x_n(s)| \, d\mu \leqslant \|x_n\| \leqslant \|x\| + \varepsilon. \end{aligned}$$

Now, we can insert these three inequalities into (3.28) to get

$$\begin{aligned} \|Tx - T(x_n)\| \leqslant & \, 3 \, \|x - x_n\| + |t(x) - t(x_n)| \, \mu(A(x)) \\ & + \varepsilon \left[\mu(A(x)) + \frac{\|x\| + \varepsilon}{t(x) - \varepsilon} \right] \end{aligned}$$

for all $n \geqslant n_\varepsilon$. Since $\mu(A(x)) < \infty$, we can let $\varepsilon \to 0$ to finish the proof of continuity of T on $L^1(\Omega, \mu)$.

If $x, y \in L^1(\Omega, \mu)$, then one can find sequences x_n and y_n of μ-integrable simple functions such that $x_n \to x$ and $y_n \to y$ in the metric of $L^1(\Omega, \mu)$. Moreover, for each n, we can embed x_n and y_n in a finite dimensional subspace of μ-integrable simple functions of the form

$$\sum_{k=1}^{r_n} \alpha_k \chi_{A_k}/\mu(A_k); \qquad \alpha_k \in \mathbf{R},$$

where $\mu(A_k) > 0$ and $A_k \cap A_j = \phi$ for $k \neq j$. Since this subspace is isometrically isomorphic with l_n^1, we can use Lemma 3.3 to get

$$\|T(x_n) - T(y_n)\| \leqslant \frac{2(r_n - 1)}{r_n} \|x_n - y_n\|.$$

By continuity of T, it follows that

$$\|Tx - Ty\| \leqslant 2 \|x - y\|,$$

i.e., $K_T(L^1(\Omega, \mu)) \leqslant 2$. On the other hand, the infinite dimensional Banach space $L^1(\Omega, \mu)$ contains the n-dimensional subspaces $l_n^1(A)$ $(n = 2, 3, ...)$ of μ-integrable simple functions x of the form

$$x = \sum_{k=1}^{n} x_k \chi_{A_k}/\mu(A_k), \qquad x_k \in \mathbf{R},$$

where $\mu(A_k) > 0$ and $A_k \cap A_j = \phi$ for $k \neq j$. Hence Lemma 3.2 yields $K_T(L^1(\Omega, \mu)) = 2$.

To show the optimality of T, let P be a selection of the metric projection $\mathscr{P}$: $L^1(\Omega, \mu) \to 2^{B_1}$. Moreover, suppose that $x \in l_n^1(A)$ is defined by

$$x = \frac{1}{n-1} \sum_{k=1}^{n} \chi_{A_k}/\mu(A_k). \tag{3.29}$$

Then we have

$$Px(s) = 0 \tag{3.30}$$

almost everywhere on $\Omega \backslash A$, where

$$A = \bigcup_{k=1}^{n} A_k.$$

Indeed, suppose that $Px(s) \neq 0$ on a measurable subset C of $\Omega \backslash A$ such that $\mu(C) > 0$. Since P is a selection of $\mathscr{P}$, we have $\|Px\| = 1$ and

$$\|x - Px\| \leqslant \|x - y\|$$

for all $y \in B_1$. Equivalently, in view of the Kolmogorov criterion [16], we have

$$\tau_x(y) := \int_Z |Px - y| \, d\mu + \int_{\Omega \backslash Z} (Px - y) \, sgn(x - Px) \, d\mu \geqslant 0$$

for all $y \in B_1$, where

$$Z = \{s \in \Omega : x(s) = Px(s)\}.$$

In particular, if $y = \chi_{\Omega \backslash C} Px$ then $\|y\| \leqslant \|Px\| = 1$ and

$$\tau_x(y) = -\int_C |Px| \, d\mu < 0.$$

This contradiction proves (3.30). Since $\|Px\| = 1$, it follows from (3.30) that $z_j \geqslant 1/n$ for some j, where $z = (z_1, ..., z_n)$, $\|z\| = 1$, and

$$z_k = \int_{A_k} |Px| \, d\mu, \qquad k = 1, ..., n.$$

Define $y^j \in l_n^1(A)$ by

$$y^j = \frac{1}{n-1} \sum_{\substack{k=1 \\ k \neq j}}^{n} \chi_{A_k} / \mu(A_k)$$

and note that

$$\|y^j\| = 1 \qquad \text{and} \qquad \|x - y^j\| = \frac{1}{n-1}. \tag{3.31}$$

Moreover, as in the proof of Lemma 3.2, we show that

$$Tx = \frac{1}{n} \sum_{k=1}^{n} \chi_{A_k} / \mu(A_k),$$

where x is defined by (3.29). Hence we get

$$\|Tx - y^j\| = \frac{2}{n} \tag{3.32}$$

and

$$\|Px-y^j\| = \int_{A_j} |Px|\, d\mu + \sum_{i \neq j} \int_{A_i} |Px-y^j|\, d\mu$$

$$\geqslant \int_{A_j} |Px|\, d\mu + \sum_{i \neq j} \left| \int_{A_i} (|Px| - |y^j|)\, d\mu \right|$$

$$= z_j + \sum_{i \neq j} \left| z_i - \frac{1}{n-1} \right|.$$

Now, we can repeat *mutatis mutandis* the second part of the proof of Theorem 3.2 to get

$$\|Px-y^j\| \geqslant \frac{2}{n}.$$

Hence it follows from (3.31) and (3.32) that

$$\|Px - Py^j\| = \|Px - y^j\| \geqslant \|Tx - Ty^j\| = \frac{2(n-1)}{n} \|x - y^j\|$$

for every selection P of $\mathscr{P}$. Since n can be arbitrarily large, we have $K_{\mathscr{P}}(L^1(\Omega, \mu)) \geqslant 2$, which completes the proof of the optimality of T. Finally, by (3.6) we have

$$T(x) = T(\alpha x + (1-\alpha)\, Tx), \qquad \alpha \geqslant 0,$$

for every μ-integrable simple function x. Since the set of all such functions is dense in $L^1(\Omega, \mu)$, we can use continuity of T to prove the sunny property for T. ∎

Recall that we have $K_R(L^1(\Omega, \mu)) = 2$ for the radial selection R of the metric projection $\mathscr{P}$: $L^1(\Omega, \mu) \to 2^{B_1}$. Clearly, R is also sunny. Therefore, it follows from Theorem 3.3 that Theorem 2.2 is not true for the infinite dimensional Banach space $L^1(\Omega, \mu)$. Finally, note that the orthogonal selection T: $l^1 \to B_1$ differs from the radial selection R by its finite dimensional behaviour. More precisely, if $x \in l^1$ and $\|x\| > 1$, then by (3.4) we have

$$Tx = (Tx_1, ..., Tx_n, 0, 0, ...),$$

where $n = \max\{k : |x_k| > t(x)\} < \infty$. However, this is not true for $Rx = x/\|x\|$ in general.

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